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Power contexts and their concept lattices

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ABSTRACT

We introduce a framework for the study of formal contexts and their lattices induced by the additional structure of self-relations on top of the traditional incidence relation. The induced contexts use subsets as objects and attributes, hence the name power context and power concept. Six types of new incidence relations are introduced by taking into account all possible combinations of universal and existential quantifiers as well as the order of the quantifications in constructing the lifted power contexts. The structure of the power concept lattice is investigated through projection mappings from the baseline objects and attributes to those of the power context, respectively. We introduce the notions of extensional consistency and intensional consistency, corresponding to the topological notions of continuity in the analogous setting when concepts are viewed as closed sets. We establish Galois connections for these notions of consistency. We further introduce the notion of faithfulness for the first type of lifted incidence relation based on the fact that it can be equivalently characterized by a concept-faithful morphism. We also present conditions under which the power concept lattice serves as a factor lattice of the base concept lattice.

0. Introduction

In [28,29], an approach was proposed to characterize and identify research communities as formal concepts. This approach suggests an ambient formal context consisting of all authors and all venues (i.e., journals and conferences): an author is related to a venue if the author ever publishes in the venue. One may then attempt to define a research community as a formal concept of the ambient context, consisting of authors who publish in venues that they share.

Further examination suggests that the issue may not be so simple. Intuitively, a research community represents a research area, such as "Information Retrieval," "Software Engineering" or "Theoretical Computer Science," as cataloged in ACM Subject Headings, AMS Classification, or NLM MeSH. If formal concepts were to be able to capture research communities, then these subject headings should correspond to concepts of the ambient context, each represented by a specific group of authors (say A) who publish in a specific group of venues (say B). Instantiating the theory of Formal Concept Analysis (FCA [26]) to this scenario, the research community represented by a pair (A, B) must have the property that all authors in A publish in all venues in B. This is counterintuitive. Suppose (A, B) represents the research community of "Theoretical Computer Science". It would be too strict to require all members of the research community of "Theoretical Computer Science" to publish in all possible Theoretical Computer Science venues. In fact, rarely does any author do this.

The insight given in [29] is that even though an individual author of a community may not publish in all venues of a subject area, it is more likely that the researcher has a co-author who publishes in additional venues. The co-authors' co-authors may cover even more venues, making it likely that the co-authorship network (the closure of the co-authorship

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relation) of a researcher as a whole covers most, if not all, of the venues of a research community. Preliminary results on using this approach to identify the Cell Cycle research community is reported in [13], where top 50 key venues of the Cell Cycle research community were identified and manually validated by a domain expert.

The "Research Community as Formal Concept" approach suggests some basic topics for FCA. This paper focuses on its theoretical ramifications, by incorporating self-relations $\rho \subseteq X \times X$ and $\theta \subseteq Y \times Y$ to a standard setup (X, Y, I), where $I \subseteq X \times Y$. In the "Research Community as Formal Concept" framework, we are interested in lifting the underlying context (X, Y, I) to a new context at the powerset level, $(X/\rho, Y/\theta, \tilde{I})$, where $X/\rho \subseteq 2^X$, $Y/\theta \subseteq 2^Y$, and $\tilde{I}$ can be constructed in several ways as illustrated in Section 2. The main motivating case of capturing the research community using power concepts amounts to the following instantiation:

- X: a set of authors,
- Y: a set of publication venues,
- $(x, y) \in I$ if x publishes in y,
- $\rho \subseteq X \times X$: the transitive, reflexive closure of a co-authorship relation,
- $\theta \subseteq Y \times Y$: the identity relation (i.e., $\theta = \mathrm{id}_Y$),
- U: the partition induced by ρ,
- $(u, y) \in \tilde{I}$, where $u \in U$ and $y \in Y$, if there exists $x \in u$ such that $(x, y) \in I$.

It is in this setup that experimental results in [13] were carried out. In fact, there exist networks amongst research venues as well. For instance, there may be related publication connections between different journals through shared editorial board members. Research venues may be (and have been) classified into different categories according to the fields they represent. These observations motivate us to develop a comprehensive framework to support the theory of "Research Community as Formal Concept".

In this paper, we introduce two self-relations ρ and θ supplied on the object set X and the attribute set Y respectively, on top of a classical context (X, Y, I). Through the notion of lifted incidence relation $\tilde{I}_k$ ($k \in \{1, \ldots, 6\}$), we establish relationships between the base concept lattice generated from (X, Y, I) and the power concept lattice generated from $(X/\rho, Y/\theta, \tilde{I}_k)$. The relationships are presented as Galois connections between the base concept lattice and its power concept lattice. In order to do so, we introduce the notions of extensional consistency and intensional consistency on lifted incidence relations, respectively. We show that each type of consistency gives rise to a Galois connection. We also introduce the notion of faithfulness according to the first type of lifted incidence relation and show that the power concept lattice can serve as a factor lattice of the base concept lattice in this case.

There are several related works in the literature. In [9], an ontology-based approach was proposed to evaluate the concept similarity using existing domain ontology in the framework of FCA. In [5], equivalence relations were supplied as a model of special forms of additional information on the objects of the original context. This approach is intended to abstract away the desired concepts which are compatible with the additional information. The problem of factorization of a concept lattice has been studied in the fuzzy setting. In [2], similarity relations on different levels in fuzzy conceptual structures were studied. In [4], the similarity relations between concepts were characterized through extents or intents and factorization patterns of the concept lattice were established corresponding to variable user-specified thresholds.

Our work is distinct from these approaches in that we explicitly introduce, in the basic setup, self-relations on objects and attributes into the classical setting of FCA. This is more generic and accommodates many special cases while at the same time lending the framework to reexamination from fuzzy or ontological analysis. Moreover, our study on how the additional self-relations influence the concept lattice structure through the connections between the base context (concept lattice) and the power context (concept lattice) provides a richer interplay between the more "concrete" and the more "abstract" approaches.

The rest of the paper is organized as follows. Section 1 recalls the basics of FCA. Section 2 introduces power contexts (concepts) and studies their basic properties. Section 3 establishes Galois connections between base and power concept lattices by introducing the notions of extensional consistency and intensional consistency. Section 4 concentrates on the first type of lifted incidence relation and identifies conditions under which a power concept lattice forms a factor lattice of its base concept lattice. The topic of research community is revisited in Section 5 in light of these theoretical developments, followed by the concluding remarks and future work in Section 6.

1. Preliminaries

We briefly recall the basics of Formal Concept Analysis (FCA). The notations and terminologies used in this paper mainly follow [11] which can be referred to for further details.

1.1. Formal concepts

The generality of FCA stems from the fact that the basic setup is a binary relation between two sets, and not merely a binary self-relation as encountered in Graph Theory. Formally, a *formal context* (briefly, *context*) is a triplet (X, Y, I), where X and Y are sets and $I \subseteq X \times Y$ a binary relation from X to Y. Elements of X and Y are called objects and attributes, respectively.

The relation I, regulating which objects have which attributes, is called the *incidence relation*. If $X_1 \subseteq X$ and $Y_1 \subseteq Y$, then $(X_1, Y_1, I \cap (X_1 \times Y_1))$ is called a *subcontext* of (X, Y, I).

For a context (X, Y, I), the operator $(\cdot)^I$ on object and attribute sets are defined as:

$$A^I := \{y \in Y \mid (x, y) \in I \text{ for all } x \in A\}$$
$$B^I := \{x \in X \mid (x, y) \in I \text{ for all } y \in B\}$$

where $A \subseteq X$ and $B \subseteq Y$. Intuitively, A^I consists of all attributes shared by all objects in A and B^I consists of all objects common to all attributes in B. For singleton sets $\{x\} \subseteq X$ and $\{y\} \subseteq Y$, we abbreviate $\{x\}^I$ as x^I and $\{y\}^I$ as y^I.

A subset pair (A, B) with $A \subseteq X, B \subseteq Y$ is called a *(formal) concept* of a given context (X, Y, I) if $A^I = B$ and $B^I = A$. A and B are called the *extent* and *intent*, respectively. A subset $A \subseteq X$ is an extent (of the attribute subset A^I) if and only if $A = A^{II}$, and dually, a subset $B \subseteq Y$ is an intent (of the object subset B^I) if and only if $B = B^{II}$. In particular, extents of the form y^I with $y \in Y$ (x^{II} with $x \in X$) are called *attribute extents* (*object extents*). Object intent and attribute intent can be defined similarly.

We use $\mathfrak{B}(X, Y, I)$ to denote the set of all concepts of (X, Y, I). The order $\leq$ on $\mathfrak{B}(X, Y, I)$ is defined by $(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 (:\Leftrightarrow B_2 \subseteq B_1)$. When the order relation holds, (A_1, B_1) is called a *subconcept* of (A_2, B_2), or equivalently, (A_2, B_2) is a *superconcept* of (A_1, B_1). We use $[(A_1, B_1), (A_2, B_2)]$ to denote the interval of concepts determined by (A_1, B_1) and (A_2, B_2), i.e., those x satisfying $(A_1, B_1) \leq x \leq (A_2, B_2)$.

Lemma 1.1 ([11]). *Let (A, B) and (A', B') be concepts of (X, Y, I) with $(A, B) \leq (A', B')$. Then*

$$[(A, B), (A', B')] = \mathfrak{B}(A', B, I \cap (A' \times B)).$$

The following *Basic Theorem of Concept Lattice* tells us that $\mathfrak{B}(X, Y, I)$ is always a complete lattice.

Theorem 1.1 ([11,26]). *The concepts $\mathfrak{B}(X, Y, I)$ of the context (X, Y, I), under the order $\leq$, is a complete lattice in which the infimum and supremum are given by*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)^{II}\right),$$
$$\bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t\right)^{II}, \bigcap_{t\in T} B_t\right).$$

As indicated in the previous theorem, the intersection of any collection of extents (intents) remains to be an extent (intent). Given a context (X, Y, I), we denote the lattice of all extents as $\mathfrak{B}_o(X, Y, I)$ and the lattice of all intents as $\mathfrak{B}_a(X, Y, I)$. Since the extent and intent of every concept uniquely determine each other, $\mathfrak{B}_o(X, Y, I)$ with set inclusion is dually order-isomorphic to $\mathfrak{B}_a(X, Y, I)$. For this reason, we sometimes only consider the extent lattice or intent lattice in our discussion. Furthermore, every concept lattice can be depicted by a line diagram with reduced labeling. A more detailed discussion about this can be referenced in [11].

1.2. Continuity and order

The notion of continuity [8,11,17,18] plays a vital role in FCA for their capability in modeling concept scaling and data abstraction. Given two contexts (X, Y, I) and (S, T, J), and two mappings $f : X \to S$ and $g : Y \to T$:

(1) f is said to be *extensionally continuous* if $f^{-1}(C)$, the preimage of C under f, is an extent of (X, Y, I) for any extent C of (S, T, J); g is said to be *intensionally continuous* if $g^{-1}(D)$ is an intent of (X, Y, I) for any intent D of (S, T, J). If f is extensionally continuous and g is intensionally continuous, then the pair (f, g) is said to be *continuous*.
(2) The pair (f, g) is said to be *incidence-preserving* if $(f(x), g(y)) \in J$ for any $x \in X$ and $y \in Y$ with $(x, y) \in I$; (f, g) is said to be *incidence-reflecting* if $(f(x), g(y)) \in J$ entails $(x, y) \in I$ for any $x \in X$ and $y \in Y$.
(3) The pair (f, g) is said to be *concept-preserving* if $(g(B)^J, f(A)^J)$ is a concept of (S, T, J) for any concept (A, B) of (X, Y, I).
(4) The pair (f, g) is said to be *concept-faithful* if it is both continuous and concept-preserving.

Theorem 1.2 ([11]). *Let (f, g) be an incidence-preserving pair from (X, Y, I) to (S, T, J). Then (f, g) is concept-faithful if and only if*

$$(s, t) \notin J \Rightarrow \exists (x, y) \notin I \text{ s.t. } f^{-1}(t^J) \subseteq y^I \text{ and } g^{-1}(s^J) \subseteq x^I$$

for any $s \in S$ and $t \in T$.

Let $f : L \to M$ be a mapping between complete lattices L and M. f is said to be $\vee$-*preserving* ($\wedge$-*preserving*) if $f(\bigvee A) = \bigvee f(A)$ ($f(\bigwedge A) = \bigwedge f(A)$) holds for any $A \subseteq L$. A *complete homomorphism* is both $\vee$-preserving and $\wedge$-preserving.

Galois connections are a widely used tool to bridge ordered structures [1,7,10] and has different formulations. Here we are only interested in the isotone form. Let $f : L \to M$ and $g : M \to L$ be mappings between partially ordered sets L and M. The pair (f, g) is said to be an *isotone Galois connection* if for any $x \in L$ and $y \in M$,

$$f(x) \leq y \quad \text{iff } x \leq g(y).$$

In subsequent discussions, the word isotone is omitted when Galois connections are referred to. When f and g are order-preserving, (f, g) forms a Galois connection if and only if the compositions fg and gf satisfy the inequalities $fg \leq \mathrm{id}_M$ and $gf \geq \mathrm{id}_L$, where the order at the mapping level is defined coordinatewise.

A basic property of a Galois connection between complete lattices is as follows.

Proposition 1.1 (*[11]*). *Let (f, g) be a Galois connection between complete lattices L and M. Then f is $\vee$-preserving and g is $\wedge$-preserving. Moreover, for any $x \in L$,*

$$f(x) = \bigwedge\{y \in M \mid x \leq g(y)\}.$$

Dually, for any $y \in M$,

$$g(y) = \bigvee\{x \in L \mid f(x) \leq y\}.$$

Let θ be a binary relation on a set X. We use $[x]_\theta$ to denote the subset of elements θ-related to x, that is, $[x]_\theta = \{y \in X \mid (x, y) \in \theta\}$. $X/\theta = \{[x]_\theta \mid x \in X\}$ denotes the set of all subsets generated in this way. For any $A \subseteq X$, let $[A]_\theta = \{[x]_\theta \mid x \in A\}$. Also, π_θ denotes the canonical projection from X to X/θ, i.e., $\pi_\theta(x) = [x]_\theta$ for any $x \in X$. For any $\alpha \subseteq X/\theta$, we use $\pi_\theta^{-1}(\alpha)$ to represent the preimage of α, i.e., $\pi_\theta^{-1}(\alpha) = \{x \in X \mid [x]_\theta \in \alpha\}$. It is easy to see that $[\pi_\theta^{-1}(\alpha)]_\theta = \alpha$. Particularly, when θ is an equivalence relation, X/θ will be the partition on X induced by θ, and π_θ the canonical quotient map from X to X/θ. In subsequent discussions, π_θ, $[A]_\theta$ and $[x]_\theta$ are abbreviated as π, $[A]$ and $[x]$ respectively when there is no confusion caused.

Remark. It should be noted that the (Polish) notation $[x]_\theta$ is traditionally reserved for equivalence relations θ. For notational convenience, we use it to represent the forward mapping induced by the relation θ without requiring it to be an equivalence relation. The "non-fix" or applicative notation $\theta(x)$ turns out to be more clumsy for our setting.

We now turn to complete congruence relations on complete lattices. An equivalence relation η on a complete lattice L is a *complete congruence relation* if it is closed with respect to both arbitrary infimum and supremum, i.e., for any subset $\{(x_t, y_t) \mid t \in T\} \subseteq \eta$ with T an index set,

$$\left(\bigwedge_{t\in T} x_t, \bigwedge_{t\in T} y_t\right) \in \eta \quad \text{and} \quad \left(\bigvee_{t\in T} x_t, \bigvee_{t\in T} y_t\right) \in \eta.$$

A complete congruence relation induces an order $\leq$ on L/η defined as

$$[x]_\eta \leq [y]_\eta :\Leftrightarrow (x, x \wedge y) \in \eta \quad (:\Leftrightarrow (x \vee y, y) \in \eta).$$

When η is a complete congruence relation on a complete lattice L, $(L/\eta, \leq)$ is a lattice and is called the *factor lattice* of L with respect to η.

As far as congruence relations and factor lattices are concerned, we always have

Theorem 1.3 (*[11,22]*). *If η is a complete congruence relation on a complete lattice L, then $x \mapsto [x]_\eta$ is a surjective complete homomorphism from L to L/η. Conversely, for any surjective complete homomorphism $f : L \to M$ between complete lattices L and M, $\ker(f) := \{(x, y) \mid x, y \in L \text{ and } f(x) = f(y)\}$ is a complete congruence relation on L. Additionally, $[x]_{\ker(f)} \mapsto f(x)$ is an order-isomorphism from $L/\ker(f)$ to M.*

2. Power context

In this section, we first introduce six types of liftings on a conventional context and then investigate fundamental properties of power contexts and power concepts.

2.1. Liftings and power contexts

The notion of power context is based on a classical context (X, Y, I) with two binary relations separately on objects and attributes, i.e., $\rho \subseteq X \times X$ and $\theta \subseteq Y \times Y$. Formally, we give the following definition.

Definition 2.1. Let (X, Y, I) be a context with binary relations $\rho \subseteq X \times X$ and $\theta \subseteq Y \times Y$. Then we call the quintuple (X, Y, I, ρ, θ) a *relationally enriched context*, (briefly, (ρ, θ)-*context*). In comparison, (X, Y, I) is called the *base context*.

Remark 2.1. In the remainder of this paper, for any (ρ, θ)-context mentioned, we require both ρ and θ to be reflexive. In this case, the (ρ, θ)-context has the non-emptiness property: for any $x \in X$ and $y \in Y$, both $[x]$ and $[y]$ are non-empty. This is a mild condition for technical convenience and readability only.

Given a (ρ, θ)-context, one can naturally induce contexts that take X/ρ the object set and Y/θ the attribute set. For this purpose, we adopt the first-order logic formalism to help exhaustively enumerate all possible constructions or "liftings," by considering combinations of universal and existential quantifiers as well as the order of the quantifications. As a result, six types of induced liftings are enumerated and captured in the following definition.

Definition 2.2. Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Six *induced contexts* $\tilde{I}_k^{(\rho,\theta)} \subseteq X/\rho \times Y/\theta$ $(k \in \{1, \dots, 6\})$ are defined as, for any $x \in X$ and $y \in Y$:

(1) $([x], [y]) \in \tilde{I}_1^{(\rho,\theta)} :\Leftrightarrow (\exists x' \in [x])(\exists y' \in [y])\ (x', y') \in I$;
(2) $([x], [y]) \in \tilde{I}_2^{(\rho,\theta)} :\Leftrightarrow (\forall x' \in [x])(\exists y' \in [y])\ (x', y') \in I$;
(3) $([x], [y]) \in \tilde{I}_3^{(\rho,\theta)} :\Leftrightarrow (\exists x' \in [x])(\forall y' \in [y])\ (x', y') \in I$;
(4) $([x], [y]) \in \tilde{I}_4^{(\rho,\theta)} :\Leftrightarrow (\forall x' \in [x])(\forall y' \in [y])\ (x', y') \in I$;
(5) $([x], [y]) \in \tilde{I}_5^{(\rho,\theta)} :\Leftrightarrow (\forall y' \in [y])(\exists x' \in [x])\ (x', y') \in I$;
(6) $([x], [y]) \in \tilde{I}_6^{(\rho,\theta)} :\Leftrightarrow (\exists y' \in [y])(\forall x' \in [x])\ (x', y') \in I$;

where $[x]$ and $[y]$ are abbreviations of $[x]_\rho$ and $[y]_\theta$, respectively.

We call these $\tilde{I}_k$'s *lifted incidence relations on* (X, Y, I, ρ, θ). Correspondingly, $(X/\rho, Y/\theta, \tilde{I}_k^{(\rho,\theta)})(k \in \{1, \dots, 6\})$ are called *power contexts of* (X, Y, I, ρ, θ).

In the sequel, when base and power object, attribute, concept, and concept lattice are cited, they are meant to denote the corresponding notions in the setting of base and power context, respectively. Meanwhile, for convenience, we simplify $\tilde{I}_k^{(\rho,\theta)}$ as $\tilde{I}_k$ whenever the notational omission of (ρ, θ) does not cause ambiguity. Moreover, we always use $\pi_\rho : X \to X/\rho$ and $\pi_\theta : Y \to Y/\theta$ to separately denote the canonical projections on object and attribute sets.

Remark 2.2. (1) Liftings in Definition 2.2 are essentially aggregation on formal contexts. Depending on the choice of constraints ρ and θ, other types of conventional preprocessing strategies in data analysis can also be induced. For example, all liftings become dimensional reductions if ρ is the identity relation on objects X, i.e., $\rho = \mathrm{id}_X$, while θ is only the identity relation on a subset B of Y, i.e., $\theta = \mathrm{id}_B$. If $\rho = \mathrm{id}_A$ and $\theta = \mathrm{id}_B$ for some $A \subseteq X$ and $B \subseteq Y$, then all liftings correspond to sampling on the base context.

(2) When represented by binary matrices, the six types of liftings can be geometrically interpreted as follows. For any $x \in X$ and $y \in Y$, the entry $([x], [y])$ receives a 1 in $\tilde{I}_1$ if and only if the rectangle $[x] \times [y]$ in I contains at least a 1; $([x], [y])$ receives a 1 in $\tilde{I}_2$ if and only if each row of the rectangle $[x] \times [y]$ in I contains at least a 1; $([x], [y])$ receives a 1 in $\tilde{I}_3$ if and only if the rectangle $[x] \times [y]$ in I contains at least a row of all 1s; $([x], [y])$ receives a 1 in $\tilde{I}_4$ if and only if the rectangle $[x] \times [y]$ in I are all 1s; $([x], [y])$ receives a 1 in $\tilde{I}_5$ if and only if each column of the rectangle $[x] \times [y]$ in I contains at least a 1; $([x], [y])$ receives a 1 in $\tilde{I}_6$ if and only if the rectangle $[x] \times [y]$ in I contains at least a column of all 1s.

According to the geometric observation in the previous remark, one can note that: for any (ρ, θ)-context, (π_ρ, π_θ) is incidence-preserving with respect to $\tilde{I}_1$; (π_ρ, π_θ) is incidence-reflecting with respect to $\tilde{I}_4$. Formally,

Proposition 2.1. *Given a (ρ, θ)-context (X, Y, I, ρ, θ), for any $x \in X$ and $y \in Y$,*

(1) $(x, y) \in I$ *implies* $([x], [y]) \in \tilde{I}_1$;
(2) $([x], [y]) \in \tilde{I}_4$ *implies* $(x, y) \in I$.

Moreover, there exists a containment hierarchy amongst the six types of lifted incidence relations which can be illustrated by Fig. 1.

Proposition 2.2. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Then*

(1) $\tilde{I}_4 \subseteq \tilde{I}_2, \tilde{I}_3, \tilde{I}_5, \tilde{I}_6 \subseteq \tilde{I}_1$; $\tilde{I}_3 \subseteq \tilde{I}_5$; $\tilde{I}_6 \subseteq \tilde{I}_2$;
(2) $(\tilde{I}_2)^{-1} = (\widetilde{I^{-1}})_5$ *and* $(\tilde{I}_3)^{-1} = (\widetilde{I^{-1}})_6$.

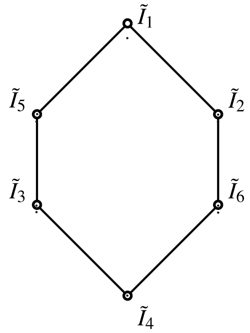

Fig. 1. Containment hierarchy of liftings for any (ρ, θ)-context.

$$\begin{tikzcd}
(X, Y, I) \arrow[r, "(\mathrm{id}_X,\ \theta)"] \arrow[d, "(\rho,\ \mathrm{id}_Y)"] \arrow[dr, "(\rho,\ \theta)"] & (X, Y/\theta, \tilde{I}_k^{(\mathrm{id}_X,\theta)}) \arrow[d, "(\rho,\ \mathrm{id}_{Y/\theta})"] \\
(X/\rho, Y, \tilde{I}_k^{(\rho,\mathrm{id}_Y)}) \arrow[r, "(\mathrm{id}_{X/\rho},\ \theta)"] & (X/\rho, Y/\theta, \tilde{I}_k^{(\rho,\theta)})
\end{tikzcd}$$

Fig. 2. Commutative diagram for sequential composition of $\tilde{I}_k^{(\rho,\theta)}$ $(k \in \{1, 4\})$.

Proposition 2.2(2) manifests the duality between liftings $\tilde{I}_2$ and $\tilde{I}_5$, and that between $\tilde{I}_3$ and $\tilde{I}_6$. Note that for any context (X, Y, I), the concept lattice $\mathfrak{B}(X, Y, I)$ is always dually order-isomorphic to $\mathfrak{B}(Y, X, I^{-1})$ of the dual context (Y, X, I^{-1}). Therefore, we are facilitated to focus on the first four types of liftings $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_3$ and $\tilde{I}_4$.

Note that when θ is the identity relation on Y, i.e., $\theta = \mathrm{id}_Y$, the lifted incidence relations $\tilde{I}_k^{(\rho,\mathrm{id}_Y)}$ $(k \in \{1, \ldots, 6\})$ can be reduced to relations from X/ρ to Y, i.e., $\tilde{I}_k^{(\rho,\mathrm{id}_Y)} \subseteq X/\rho \times Y$. Dually, when ρ is the identity relation on X, i.e., $\rho = \mathrm{id}_X$, the lifted incidence relations $\tilde{I}_k^{(\mathrm{id}_X,\theta)}$ $(k \in \{1, \ldots, 6\})$ can be reduced to relations from X to Y/θ, i.e., $\tilde{I}_k^{(\mathrm{id}_X,\theta)} \subseteq X \times Y/\theta$. Moreover, liftings of different types come to be the same in case of identity constraints on objects or attributes.

Proposition 2.3. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Then*

(1) *if θ is the identity relation on Y, i.e., $\theta = \mathrm{id}_Y$, then $\tilde{I}_1^{(\rho,\mathrm{id}_Y)} = \tilde{I}_3^{(\rho,\mathrm{id}_Y)} = \tilde{I}_5^{(\rho,\mathrm{id}_Y)}$ and $\tilde{I}_2^{(\rho,\mathrm{id}_Y)} = \tilde{I}_4^{(\rho,\mathrm{id}_Y)} = \tilde{I}_6^{(\rho,\mathrm{id}_Y)}$;*
(2) *if ρ is the identity relation on X, i.e., $\rho = \mathrm{id}_X$, then $\tilde{I}_1^{(\mathrm{id}_X,\theta)} = \tilde{I}_2^{(\mathrm{id}_X,\theta)} = \tilde{I}_6^{(\mathrm{id}_X,\theta)}$ and $\tilde{I}_3^{(\mathrm{id}_X,\theta)} = \tilde{I}_4^{(\mathrm{id}_X,\theta)} = \tilde{I}_5^{(\mathrm{id}_X,\theta)}$;*
(3) *if $\rho = \mathrm{id}_X$ and $\theta = \mathrm{id}_Y$, all the six types of lifted incidence relations coincide. In this case, the lifting leads to the same context with the base context.*

There is an associative and thereby more efficient way than the naive one arising directly from the definition of liftings to fulfill the computation on $\tilde{I}_1^{(\rho,\theta)}$ and $\tilde{I}_4^{(\rho,\theta)}$.

Proposition 2.4. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Then*

$$\tilde{I}_1^{(\rho,\theta)} = \widetilde{(\tilde{I}_1^{(\rho,\mathrm{id}_Y)})}_1^{(\mathrm{id}_{X/\rho},\theta)} = \widetilde{(\tilde{I}_1^{(\mathrm{id}_X,\theta)})}_1^{(\rho,\mathrm{id}_{Y/\theta})},$$
$$\tilde{I}_4^{(\rho,\theta)} = \widetilde{(\tilde{I}_4^{(\rho,\mathrm{id}_Y)})}_4^{(\mathrm{id}_{X/\rho},\theta)} = \widetilde{(\tilde{I}_4^{(\mathrm{id}_X,\theta)})}_4^{(\rho,\mathrm{id}_{Y/\theta})}.$$

Proof. It is easy to verify that $\widetilde{(\tilde{I}_k^{(\rho,\mathrm{id}_Y)})}_k^{(\mathrm{id}_{X/\rho},\theta)} = \widetilde{(\tilde{I}_k^{(\mathrm{id}_X,\theta)})}_k^{(\rho,\mathrm{id}_{Y/\theta})}$ always holds for $k \in \{1, 4\}$. Suppose $x \in X$ and $y \in Y$. For $k = 1$, we have $([x], [y]) \in \tilde{I}_1^{(\rho,\theta)}$ iff $(x', y') \in I$ for some $x' \in [x]$ and $y' \in [y]$ iff $([x], y') \in \tilde{I}_1^{(\rho,id_Y)}$ for some $y' \in [y]$ iff $([x], [y]) \in \widetilde{(\tilde{I}_1^{(\rho,\mathrm{id}_Y)})}_1^{(\mathrm{id}_{X/\rho},\theta)}$. Therefore, $\tilde{I}_1^{(\rho,\theta)} = \widetilde{(\tilde{I}_1^{(\rho,\mathrm{id}_Y)})}_1^{(\mathrm{id}_{X/\rho},\theta)}$. For $k = 4$, we have $([x], [y]) \in \tilde{I}_4^{(\rho,\theta)}$ iff $(x', y') \in I$ for all $x' \in [x]$ and $y' \in [y]$ iff $([x], y') \in \tilde{I}_4^{(\rho,id_Y)}$ for all $y' \in [y]$ iff $([x], [y]) \in \widetilde{(\tilde{I}_4^{(\rho,\mathrm{id}_Y)})}_4^{(\mathrm{id}_{X/\rho},\theta)}$. Therefore, $\tilde{I}_4^{(\rho,\theta)} = \widetilde{(\tilde{I}_4^{(\rho,\mathrm{id}_Y)})}_4^{(\mathrm{id}_{X/\rho},\theta)}$. □

The associative equation in Proposition 2.4 is illustrated in Fig. 2. Intuitively, in order to obtain a complete lifting $\tilde{I}_k^{(\rho,\theta)}(k \in \{1, 4\})$, we can fulfill the lifting process solely on objects (attributes), followed by a lifting process solely on attributes (correspondingly, objects). However, this result does not necessarily hold for other types of liftings.

Example 2.1. Consider the base context (X, Y, I) represented by the table on the left hand side of Fig. 3. The relation ρ on X is represented as $[x_1] = \{x_1, x_2\}$, $[x_2] = \{x_2\}$, and $[x_3] = \{x_3\}$. The relation θ on Y is represented as $[y_1] = \{y_1, y_2\}$, $[y_2] = \{y_2\}$,

	y_1	y_2	y_3
x_1	×		×
x_2		×	
x_3	×	×	

$\longrightarrow$

	$[y_1]$	$[y_2]$	$[y_3]$
$[x_1]$		×	×
$[x_2]$		×	
$[x_3]$	×	×	

Fig. 3. The base and power contexts from Example 2.1.

	y_1	y_2	y_3	y_4
x_1	×			×
x_2		×		
x_3		×	×	
x_4		×		×

$\longrightarrow$

	y_1	y_2	y_3	y_4
$[x_1]$	×			×
$[x_2]$		×		×
$[x_3]$		×	×	
$[x_4]$		×	×	×

Fig. 4. The base and power contexts from Example 2.2.

and $[y_3] = \{y_3\}$. The table on the right hand side of Fig. 3 represents the power context with respect to $\tilde{I}_3^{(\rho,\theta)}$. One can check that $([x_1], [y_1]) \notin \tilde{I}_3^{(\rho,\theta)}$ and $([x_1], [y_1]) \notin (\widetilde{\tilde{I}_3^{(\mathrm{id}_X,\theta)}})_3^{(\rho,\mathrm{id}_{Y/\theta})}$, but $([x_1], [y_1]) \in (\widetilde{\tilde{I}_3^{(\rho,\mathrm{id}_Y)}})_3^{(\mathrm{id}_{X/\rho},\theta)}$. In addition, for the same (ρ, θ)-context, one can check that $([x_1], [y_1]) \in \tilde{I}_2^{(\rho,\theta)}$ and $([x_1], [y_1]) \in (\widetilde{\tilde{I}_2^{(\mathrm{id}_X,\theta)}})_2^{(\rho,\mathrm{id}_{Y/\theta})}$, but $([x_1], [y_1]) \notin (\widetilde{\tilde{I}_2^{(\rho,\mathrm{id}_Y)}})_2^{(\mathrm{id}_{X/\rho},\theta)}$.

2.2. Power concepts

Now we consider the impact of lifting operations on the cardinality of concept lattices. We show that the fourth type of lifting operation decreases the size of concept lattices. To this end, we first give the following lemma.

Lemma 2.1. *Given a (ρ, θ)-context (X, Y, I, ρ, θ), for any $x \in X$ and $y \in Y$,*

(1) $[x]^{\tilde{I}_1^{(\rho,\mathrm{id}_Y)}} = \bigcup_{a\in[x]} a^I$ *and* $[y]^{\tilde{I}_1^{(\mathrm{id}_X,\theta)}} = \bigcup_{b\in[y]} b^I$;

(2) $[x]^{\tilde{I}_4^{(\rho,\mathrm{id}_Y)}} = \bigcap_{a\in[x]} a^I$ *and* $[y]^{\tilde{I}_4^{(\mathrm{id}_X,\theta)}} = \bigcap_{b\in[y]} b^I$.

Proof. (1): Suppose $x \in X$. Then for any $d \in Y$, according to the definition of liftings, $d \in [x]^{\tilde{I}_1^{(\rho,\mathrm{id}_Y)}}$ iff $(a, d) \in I$ for some $a \in [x]$ iff $d \in a^I$ for some $a \in [x]$ iff $d \in \bigcup_{a\in[x]} a^I$. Thus $[x]^{\tilde{I}_1^{(\rho,\mathrm{id}_Y)}} = \bigcup_{a\in[x]} a^I$. The second equation can be similarly proved.

(2): Suppose $x \in X$. Then for any $d \in Y$, according to the definition of liftings, $d \in [x]^{\tilde{I}_4^{(\rho,\mathrm{id}_Y)}}$ iff $(a, d) \in I$ for any $a \in [x]$ iff $d \in a^I$ for any $a \in [x]$. Therefore, $[x]^{\tilde{I}_4^{(\rho,\mathrm{id}_Y)}} = \bigcap_{a\in[x]} a^I$. Similarly, we have the second equation. □

By Lemma 2.1(2), each object intent of $(X/\rho, Y, \tilde{I}_4^{(\rho,\mathrm{id}_Y)})$ is an intent of the base context (X, Y, I), and each attribute extent of $(X, Y/\theta, \tilde{I}_4^{(\mathrm{id}_X,\theta)})$ is an extent of (X, Y, I). Note that every extent (intent) of a given context can be generated by intersecting some attribute extents (object intents, correspondingly), and so we have the following cardinality result for the fourth type of lifting.

Proposition 2.5. *For any (ρ, θ)-context (X, Y, I, ρ, θ), we have $|\mathfrak{B}(X/\rho, Y, \tilde{I}_4^{(\rho,\mathrm{id}_Y)})| \leq |\mathfrak{B}(X, Y, I)|$, $|\mathfrak{B}(X, Y/\theta, \tilde{I}_4^{(\mathrm{id}_X,\theta)})| \leq |\mathfrak{B}(X, Y, I)|$ and $|\mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_4^{(\rho,\theta)})| \leq |\mathfrak{B}(X, Y, I)|$.*

The cardinality result in Proposition 2.5 might not hold for any other type of liftings. In this regard, we present an example for the first type of lifting to show that the power concept lattice might be more complex than the base one.

Example 2.2. Consider the base context (X, Y, I) represented by the table on the left hand side of Fig. 4. The relation ρ is a symmetric relation on X such that $[x_1] = \{x_1\}$, $[x_2] = \{x_2, x_4\}$, $[x_3] = \{x_3\}$ and $[x_4] = \{x_3, x_4\}$, and the relation θ on Y is id_Y. The table on the right hand side of Fig. 4 represents the power context with respect to $\tilde{I}_1^{(\rho,\mathrm{id}_Y)}$. Fig. 5 gives the diagrams of the base concept lattice and the power concept lattice. To be more clear, only reduced labeling of attributes are displayed in the line diagrams. One can see that the power context has one more concept which is represented by a solid node than the base one. Actually, one can observe that the base concept lattice can be order-embedded into the power concept lattice.

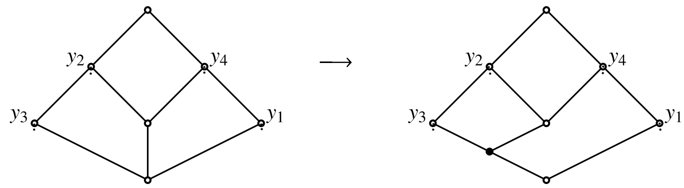

Fig. 5. The base and power concept lattices from Example 2.2.

In what follows, we will investigate the fundamental properties of power concepts with respect to the first lifting and the fourth one. Our approach focuses on the study of the relationship between base concepts and power concepts. We firstly give the following basic propositions.

Proposition 2.6. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. For any subsets $A \subseteq X$ and $B \subseteq Y$,*

(1) $[A]^{\tilde{I}_4} \subseteq [A^I] \subseteq [A]^{\tilde{I}_1}$;
(2) $[B]^{\tilde{I}_4} \subseteq [B^I] \subseteq [B]^{\tilde{I}_1}$.

Proof. We first verify (1). Suppose $A \neq \emptyset$. For any $y \in Y$ with $[y] \in [A]^{\tilde{I}_4}$, we have $([x], [y]) \in \tilde{I}_4$ for any $x \in A$. It follows that $(x, y) \in I$ for any $x \in A$ and so $y \in A^I$. Then $[y] \in [A^I]$. Next, for any $y \in Y$ with $[y] \in [A^I]$, there exists $y' \in A^I$ such that $[y] = [y']$. It is easy to verify that $[y'] \in [A]^{\tilde{I}_1}$ and it follows that $[y] \in [A]^{\tilde{I}_1}$. If $A = \emptyset$, it is clear that $[A]^{\tilde{I}_4} = [A^I] = [A]^{\tilde{I}_1} = Y/\theta$. Therefore, we have $[A]^{\tilde{I}_4} \subseteq [A^I] \subseteq [A]^{\tilde{I}_1}$. (2) can be similarly verified. □

Proposition 2.7. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. For any subsets $\alpha \subseteq X/\rho$ and $\beta \subseteq Y/\theta$,*

(1) $\pi_\theta^{-1}(\alpha^{\tilde{I}_4}) \subseteq \pi_\rho^{-1}(\alpha)^{\mathrm{I}} \subseteq \pi_\theta^{-1}(\alpha^{\tilde{I}_1})$;
(2) $\pi_\rho^{-1}(\beta^{\tilde{I}_4}) \subseteq \pi_\theta^{-1}(\beta)^{\mathrm{I}} \subseteq \pi_\rho^{-1}(\beta^{\tilde{I}_1})$.

Proof. Since arguments on (1) and (2) are very similar, we only show (1). Suppose $\alpha \neq \emptyset$. For any $y \in \pi_\theta^{-1}(\alpha^{\tilde{I}_4})$, we have $[y] \in \alpha^{\tilde{I}_4}$. This means that $([x], [y]) \in \tilde{I}_4$ for any $x \in X$ with $[x] \in \alpha$. From the definition of $\tilde{I}_4$, we have $(x, y) \in I$ for any $x \in \pi_\rho^{-1}(\alpha)$. Thus $y \in \pi_\rho^{-1}(\alpha)^{\mathrm{I}}$. Next, for any $y \in \pi_\rho^{-1}(\alpha)^{\mathrm{I}}$, we have $(x, y) \in I$ for any $x \in X$ with $[x] \in \alpha$, which implies that $([x], [y]) \in \tilde{I}_1$ for any $x \in X$ with $[x] \in \alpha$. It follows that $[y] \in \alpha^{\tilde{I}_1}$ and thus $y \in \pi_\theta^{-1}(\alpha^{\tilde{I}_1})$. If $\alpha = \emptyset$, it is clear that $\pi_\theta^{-1}(\alpha^{\tilde{I}_4}) = \pi_\rho^{-1}(\alpha)^{\mathrm{I}} = \pi_\theta^{-1}(\alpha^{\tilde{I}_1}) = Y$. Therefore, we have $\pi_\theta^{-1}(\alpha^{\tilde{I}_4}) \subseteq \pi_\rho^{-1}(\alpha)^{\mathrm{I}} \subseteq \pi_\theta^{-1}(\alpha^{\tilde{I}_1})$. □

Corollary 2.1. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. For any $(A, B) \in \mathfrak{B}(X, Y, I)$, $(\alpha, \beta) \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$ and $(\alpha', \beta') \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_4)$,*

(1) $([A]^{\tilde{I}_1\tilde{I}_1}, [A]^{\tilde{I}_1}) \leq ([B]^{\tilde{I}_1}, [B]^{\tilde{I}_1\tilde{I}_1})$;
(2) $([B]^{\tilde{I}_4}, [B]^{\tilde{I}_4\tilde{I}_4}) \leq ([A]^{\tilde{I}_4\tilde{I}_4}, [A]^{\tilde{I}_4})$;
(3) $(\pi_\theta^{-1}(\beta)^{\mathrm{I}}, \pi_\theta^{-1}(\beta)^{\mathrm{II}}) \leq (\pi_\rho^{-1}(\alpha)^{\mathrm{II}}, \pi_\rho^{-1}(\alpha)^{\mathrm{I}})$;
(4) $(\pi_\rho^{-1}(\alpha')^{\mathrm{II}}, \pi_\rho^{-1}(\alpha')^{\mathrm{I}}) \leq (\pi_\theta^{-1}(\beta')^{\mathrm{I}}, \pi_\theta^{-1}(\beta')^{\mathrm{II}})$.

From Corollary 2.1, we observe that each base concept (A, B) can induce a power concept interval on $(X/\rho, Y/\theta, \tilde{I}_1)$ and $(X/\rho, Y/\theta, \tilde{I}_4)$ separately, and dually, each power concept of $(X/\rho, Y/\theta, \tilde{I}_1)$ and $(X/\rho, Y/\theta, \tilde{I}_4)$ can induce a base concept interval separately. A natural question is: what are the appropriate conditions under which these intervals collapse to singletons? In this regard, we present the following proposition.

Proposition 2.8. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. For any $(A, B) \in \mathfrak{B}(X, Y, I)$ and $(\alpha, \beta) \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$,*

(1) $[A]^{\tilde{I}_4\tilde{I}_4} = [B]^{\tilde{I}_4} \Leftrightarrow [A]^{\tilde{I}_4} = [B]$;
(2) $\pi_\rho^{-1}(\alpha)^{\mathrm{II}} = \pi_\theta^{-1}(\beta)^{\mathrm{I}} \Leftrightarrow \pi_\rho^{-1}(\alpha)^{\mathrm{I}} = \pi_\theta^{-1}(\beta)$.

Proof. We first prove (1). Suppose $[A]^{\tilde{I}_4\tilde{I}_4} = [B]^{\tilde{I}_4}$, then $[B] \subseteq [B]^{\tilde{I}_4\tilde{I}_4} = [A]^{\tilde{I}_4\tilde{I}_4\tilde{I}_4} = [A]^{\tilde{I}_4}$. In addition, as $[A]^{\tilde{I}_4} \subseteq [B]$ by Proposition 2.6(1), we thus have $[A]^{\tilde{I}_4} = [B]$. Obviously, the reverse direction holds. Likewise, we can verify (2). □

The following result immediately follows from Lemma 1.1 and Corollary 2.1. It connects the lifting constructions with the subcontext construction given by Meschke [20].

Proposition 2.9. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Then*

(1) *for each base concept $(A, B) \in \mathfrak{B}(X, Y, I)$, there exist subcontexts $(S, T, \tilde{I}_1 \cap (S \times T))$ of $(X/\rho, Y/\theta, \tilde{I}_1)$ and $(S', T', \tilde{I}_4 \cap (S' \times T'))$ of $(X/\rho, Y/\theta, \tilde{I}_4)$ such that*

$$[([A]^{\tilde{I}_1\tilde{I}_1}, [A]^{\tilde{I}_1}), ([B]^{\tilde{I}_1}, [B]^{\tilde{I}_1\tilde{I}_1})] \cong \mathfrak{B}(S, T, \tilde{I}_1 \cap (S \times T)),$$
$$[([B]^{\tilde{I}_4}, [B]^{\tilde{I}_4\tilde{I}_4}), ([A]^{\tilde{I}_4\tilde{I}_4}, [A]^{\tilde{I}_4})] \cong \mathfrak{B}(S', T', \tilde{I}_4 \cap (S' \times T'));$$

(2) *for each power concept $(\alpha, \beta) \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$ and $(\alpha', \beta') \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_4)$, there correspondingly exist subcontexts $(H, N, I \cap (H \times N))$ and $(H', N', I \cap (H' \times N'))$ of (X, Y, I) such that*

$$[(\pi_\theta^{-1}(\beta)^{\mathrm{I}}, \pi_\theta^{-1}(\beta)^{\mathrm{II}}), (\pi_\rho^{-1}(\alpha)^{\mathrm{II}}, \pi_\rho^{-1}(\alpha)^{\mathrm{I}})] \cong \mathfrak{B}(H, N, I \cap (H \times N)),$$
$$[(\pi_\rho^{-1}(\alpha')^{\mathrm{II}}, \pi_\rho^{-1}(\alpha')^{\mathrm{I}}), (\pi^{-1}(\beta')^{\mathrm{I}}, \pi^{-1}(\beta')^{\mathrm{II}})] \cong \mathfrak{B}(H', N', I \cap (H' \times N')).$$

For a complete lattice L, let $(\mathcal{I}_L, \sqsubseteq)$ denote the interval lattice of L where $\mathcal{I}_L = \{[x, y] \mid x, y \in L \text{ and } x \leq y\}$ and the partial order $\sqsubseteq$ on $\mathcal{I}_L$ is defined as $[x, y] \sqsubseteq [x', y'] :\Leftrightarrow x \leq x'$ and $y \leq y'$. Given any (ρ, θ)-context, we can define a mapping $e : \mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_1) \to \mathcal{I}_{\mathfrak{B}_o(X,Y,I)}$ by

$$e(\alpha) := [\pi_\theta^{-1}(\beta)^{\mathrm{I}}, \pi_\rho^{-1}(\alpha)^{\mathrm{II}}],$$

where $\beta = \alpha^{\tilde{I}_1}$.

From Corollary 2.1, e is well defined. It is easy to observe that e is order-preserving. Furthermore, the following proposition shows that the image of e covers the base extent lattice $\mathfrak{B}_o(X, Y, I)$.

Proposition 2.10. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. Then for any extent A of (X, Y, I), there exists an extent α of $(X/\rho, Y/\theta, \tilde{I}_1)$ such that $A \in e(\alpha)$.*

Proof. For any extent A of (X, Y, I), let $\alpha = [A]^{\tilde{I}_1\tilde{I}_1}$. Then $A \in e(\alpha)$ holds. In fact, since $A \subseteq \pi_\rho^{-1}([A]^{\tilde{I}_1\tilde{I}_1}) = \pi_\rho^{-1}(\alpha)$, we have $A = A^{\mathrm{II}} \subseteq \pi_\rho^{-1}(\alpha)^{\mathrm{II}}$. On the other hand, let $\beta = \alpha^{\tilde{I}_1}$, then $A^I \subseteq \pi_\theta^{-1}([A^I]) \subseteq \pi_\theta^{-1}([A]^{\tilde{I}_1}) = \pi_\theta^{-1}(\alpha^{\tilde{I}_1}) = \pi_\theta^{-1}(\beta)$ by Proposition 2.6(1), which follows that $\pi_\theta^{-1}(\beta)^{\mathrm{I}} \subseteq A^{\mathrm{II}} = A$. Thus, $A \in [\pi_\theta^{-1}(\beta)^{\mathrm{I}}, \pi_\rho^{-1}(\alpha)^{\mathrm{II}}] = e(\alpha)$. □

3. Consistency

In order theory, the Galois connection is a useful tool to study the relationship between order structures. We start this section with investigating how Galois connections exist between base concept lattices and their power concept lattices. In what follows, we use k to denote an integer in the set $\{1, 2, 3, 4\}$.

Definition 3.1. Given a (ρ, θ)-context (X, Y, I, ρ, θ),

(1) $\tilde{I}_k$ is said to be *extensionally consistent* if π_ρ is extensionally continuous;
(2) $\tilde{I}_k$ is said to be *intensionally consistent* if π_θ is intensionally continuous.

According to Definition 3.1, we know that $\tilde{I}_k$ is extensionally consistent if and only if the preimage $\pi_\rho^{-1}(\alpha)$ is an extent of (X, Y, I) for any extent α of $(X/\rho, Y/\theta, \tilde{I}_k)$. Dually, $\tilde{I}_k$ is intensionally consistent if and only if the preimage $\pi_\theta^{-1}(\beta)$ is an intent of (X, Y, I) for any intent β of $(X/\rho, Y/\theta, \tilde{I}_k)$.

Proposition 3.1. *Given a (ρ, θ)-context (X, Y, I, ρ, θ), the following are equivalent:*

(1) *$\tilde{I}_k$ is extensionally consistent.*
(2) *$\pi_\rho^{-1}([y]^{\tilde{I}_k})$ is an extent of (X, Y, I) for any $y \in Y$.*
(3) *For any $x \in X$ and $y \in Y$, $\pi_\rho^{-1}([y]^{\tilde{I}_k})^I \subseteq x^I$ entails $([x], [y]) \in \tilde{I}_k$.*

Proof. (1) $\Leftrightarrow$ (2): (2) follows immediately from (1) because $[y]^{\tilde{I}_k}$ is an extent of $(X/\rho, Y/\theta, \tilde{I}_k)$ for any $y \in Y$. On the other hand, suppose α is an extent of $(X/\rho, Y/\theta, \tilde{I}_k)$. Then there exists a subset $B \subseteq Y$ such that $\pi_\rho^{-1}(\alpha) = \pi_\rho^{-1}(\bigcap_{y\in B}[y]^{\tilde{I}_k}) = \bigcap_{y\in B} \pi_\rho^{-1}([y]^{\tilde{I}_k})$. By (2), we have $\pi_\rho^{-1}(\alpha)$ is an extent of (X, Y, I).

(2) $\Leftrightarrow$ (3): Suppose $y \in Y$. We have: $\pi_\rho^{-1}([y]^{\tilde{I}_k})$ is an extent of (X, Y, I) iff $\pi_\rho^{-1}([y]^{\tilde{I}_k}) = \pi_\rho^{-1}([y]^{\tilde{I}_k})^{\mathrm{II}}$ iff $\pi_\rho^{-1}([y]^{\tilde{I}_k})^{\mathrm{II}} \subseteq \pi_\rho^{-1}([y]^{\tilde{I}_k})$ iff for any $x \in X$, $x \in \pi_\rho^{-1}([y]^{\tilde{I}_k})^{\mathrm{II}}$ entails $x \in \pi_\rho^{-1}([y]^{\tilde{I}_k})$ iff for any $x \in X$, $\pi_\rho^{-1}([y]^{\tilde{I}_k})^I \subseteq x^{\mathrm{I}}$ entails $([x], [y]) \in \tilde{I}_k$. □

When $\tilde{I}_k$ is extensionally consistent, one can naturally define a mapping

$$f_o : \mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_k) \to \mathfrak{B}_o(X, Y, I), \quad \alpha \longmapsto \pi_\rho^{-1}(\alpha).$$

Apparently, f_o is order-preserving and infimum preserving, and thus owns a dual adjoint which is proved to be

$$g_o : \mathfrak{B}_o(X, Y, I) \to \mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_k), \quad A \longmapsto [A]^{\tilde{I}_k\tilde{I}_k}.$$

Theorem 3.1. *If $\tilde{I}_k$ is extensionally consistent, then (g_o, f_o) is a Galois connection. In particular, $f_og_o \geq \mathrm{id}_{\mathfrak{B}_o(X,Y,I)}$ and $g_of_o = \mathrm{id}_{\mathfrak{B}_o(X/\rho,Y/\theta,\tilde{I}_k)}$.*

Proof. For any extent A of (X, Y, I), we have $f_og_o(A) = f_o([A]^{\tilde{I}_k\tilde{I}_k}) = \pi_\rho^{-1}([A]^{\tilde{I}_k\tilde{I}_k}) \supseteq A$, which means $f_og_o \geq \mathrm{id}_{\mathfrak{B}_o(X,Y,I)}$. On the other hand, for any extent α of $(X/\rho, Y/\theta, \tilde{I}_k)$, $g_of_o(\alpha) = g_o(\pi_\rho^{-1}(\alpha)) = [\pi_\rho^{-1}(\alpha)]^{\tilde{I}_k\tilde{I}_k} = \alpha^{\tilde{I}_k\tilde{I}_k} = \alpha$. Thus $g_of_o = \mathrm{id}_{\mathfrak{B}_o(X/\rho,Y/\theta,\tilde{I}_k)}$. Since both g_o and f_o are order-preserving, (g_o, f_o) is a Galois connection. $\square$

From Proposition 1.1 and Theorem 3.1, we have

Corollary 3.1. *If $\tilde{I}_k$ is extensionally consistent, then*

(1) *for any non-empty family $\{A_t \mid t \in T\}$ of extents of (X, Y, I),*

$$\left[\left(\bigcup_{t\in T} A_t\right)^{\mathrm{II}}\right]^{\tilde{I}_k\tilde{I}_k} = \left(\bigcup_{t\in T} [A_t]^{\tilde{I}_k\tilde{I}_k}\right)^{\tilde{I}_k\tilde{I}_k};$$

(2) *for any extent $\alpha \in \mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_k)$, $\pi_\rho^{-1}(\alpha) = (\bigcup \mathcal{H}_\alpha)^{\mathrm{II}}$, where $\mathcal{H}_\alpha = \{A \mid A \in \mathfrak{B}_o(X, Y, I)$ and $\alpha^{\tilde{I}_k} \subseteq [A]^{\tilde{I}_k}\}$.*

Apparently, all the above results valid to extensional consistency have their counterparts for intensional consistency. We enumerate them without any detailed proof.

Proposition 3.2. *Given a (ρ, θ)-context (X, Y, I, ρ, θ), the following are equivalent:*

(1) *$\tilde{I}_k$ is intensionally consistent.*
(2) *$\pi_\theta^{-1}([x]^{\tilde{I}_k})$ is an intent of (X, Y, I) for any $x \in X$.*
(3) *For any $x \in X$ and $y \in Y$, $\pi_\theta^{-1}([x]^{\tilde{I}_k})^I \subseteq y^I$ entails $([x], [y]) \in \tilde{I}_k$.*

Theorem 3.2. *If $\tilde{I}_k$ is intensionally consistent, then the pair of mappings (g_a, f_a), where*

$$f_a : \mathfrak{B}_a(X/\rho, Y/\theta, \tilde{I}_k) \to \mathfrak{B}_a(X, Y, I), \quad \beta \longmapsto \pi_\theta^{-1}(\beta)$$
$$g_a : \mathfrak{B}_a(X, Y, I) \to \mathfrak{B}_a(X/\rho, Y/\theta, \tilde{I}_k), \quad B \longmapsto [B]^{\tilde{I}_k\tilde{I}_k}$$

forms a Galois connection. In particular, $f_ag_a \geq \mathrm{id}_{\mathfrak{B}_a(X,Y,I)}$ and $g_af_a = \mathrm{id}_{\mathfrak{B}_a(X/\rho,Y/\theta,\tilde{I}_k)}$.

Corollary 3.2. *Let $\tilde{I}_k$ be intensionally consistent.*

(1) *For any non-empty family $\{B_t \mid t \in T\}$ of intents of (X, Y, I),*

$$\left[\left(\bigcup_{t\in T} B_t\right)^{\mathrm{II}}\right]^{\tilde{I}_k\tilde{I}_k} = \left(\bigcup_{t\in T} [B_t]^{\tilde{I}_k\tilde{I}_k}\right)^{\tilde{I}_k\tilde{I}_k}.$$

(2) *For any intent $\beta \in \mathfrak{B}_a(X/\rho, Y/\theta, \tilde{I}_k)$, let $\mathcal{F}_\beta = \{B \mid B \in \mathfrak{B}_a(X, Y, I)$ and $\beta^{\tilde{I}_k} \subseteq [B]^{\tilde{I}_k}\}$. Then $\pi_\theta^{-1}(\beta) = (\bigcup \mathcal{F}_\beta)^{\mathrm{II}}$.*

4. Faithfulness

As discussed in Section 3, Galois connections can be smoothly established between base concept lattices and power concept lattices when extensional or intensional consistency is met. In practice, it is also attractive to explore the factorization relation between power concept lattices and their corresponding base concept lattices: if the power concept lattice turns out to be a factor lattice of the base concept lattice, then the latter structure can be faithfully reflected by the former one. In this section, we only focus on the first type of lifted incidence relation which corresponds to the motivating case study of research communities mentioned in Introduction. We start our discussion about this factorization problem by formulating the notion of faithfulness.

	y_1	y_2	y_3
x_1		×	
x_2			×
x_3	×	×	
x_4	×		×

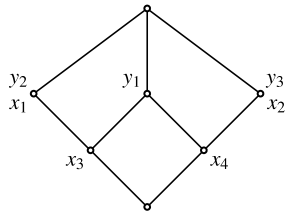

Fig. 6. The base context (left) and the base concept lattice (right) from Example 4.1.

	y_1	y_2	y_3
e_1	×	×	
e_2	×		×

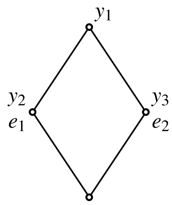

Fig. 7. The power context (left) and the power concept lattice (right) from Example 4.1.

	y_1	y_2	y_3
x_1		×	
x_2			×
x_3	×	×	×
x_4		×	×

$\longrightarrow$

	y_1	y_2	y_3
e_1		×	×
e_2	×	×	×

Fig. 8. The base context (left) and the power context (right) from Example 4.2.

Definition 4.1. Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. $\tilde{I}_1$ is said to be *faithful* if for any $x \in X$ and $y \in Y$, $\pi_\theta^{-1}([x]^{\tilde{I}_1})^I \times \pi_\rho^{-1}([y]^{\tilde{I}_1})^I \subseteq I$ entails $([x], [y]) \in \tilde{I}_1$.

In other words, $\tilde{I}_1$ is faithful if and only if whenever a power object $[x]$ is not $\tilde{I}_1$-related to a power attribute $[y]$, there must exist a base object $x' \in X$ and a base attribute $y' \in Y$ such that $x' \in \pi_\theta^{-1}([x]^{\tilde{I}_1})^I$, $y' \in \pi_\rho^{-1}([y]^{\tilde{I}_1})^I$, but $(x', y') \notin I$.

It is easy to see that if $\rho = \mathrm{id}_X$ and $\theta = \mathrm{id}_Y$, $\tilde{I}_1$ is always faithful. In this case, the condition described in Definition 4.1 is reduced to: for any $x \in X$ and $y \in Y$, $x^{II} \times y^{II} \subseteq I$ entails $(x, y) \in I$, which is always true in the base context since $x \in x^{II}$ and $y \in y^{II}$.

To make the notion of faithfulness more clear, we present the following example.

Example 4.1. Consider the base context represented by the table on the left hand of Fig. 6. The partition on X corresponding to the equivalence relation ρ is $X/\rho = \{e_1, e_2\}$, where $e_1 = \{x_1, x_3\}$, $e_2 = \{x_2, x_4\}$. The equivalence relation on Y is id_Y. The power context and its concept lattice are displayed in Fig. 7. To verify the faithfulness of $\tilde{I}_1$, we only need to check if both (e_1, y_3) and (e_2, y_2) meet Definition 4.1: for $(e_1, y_3) \notin \tilde{I}_1$, we have $x_3 \in e_1^{\tilde{I}_1 I}$ and $y_3 \in \pi_\rho^{-1}(y_3^{\tilde{I}_1})^{\mathrm{I}}$ such that $(x_3, y_3) \notin I$; for $(e_2, y_2) \notin \tilde{\mathrm{I}}_1$, we have $x_4 \in e_2^{\tilde{I}_1 I}$ and $y_2 \in \pi_\rho^{-1}(y_2^{\tilde{I}_1})^I$ such that $(x_4, y_2) \notin I$.

It is easy to see that the pair (π_ρ, π_θ) is incidence-preserving from (X, Y, I) to $(X/\rho, Y/\theta, \tilde{I}_1)$. By Theorem 1.2, we have

Proposition 4.1. *For any (ρ, θ)-context (X, Y, I, ρ, θ), $\tilde{\mathrm{I}}_1$ is faithful if and only if (π_ρ, π_θ) is concept-faithful.*

It is clear to see that if $\tilde{I}_1$ is faithful, then it is both extensionally and intensionally consistent. An interesting question is whether the faithfulness of a lifted incidence relation can be inferred solely by extensional and intensional consistency. As illustrated by the following example, this is not true.

Example 4.2. Consider the base context given in Fig. 8. The partition on X corresponding to ρ is $X/\rho = \{e_1, e_2\}$, where $e_1 = \{x_1, x_2\}$, $e_2 = \{x_3, x_4\}$. It is easy to verify that $\tilde{I}_1$ is extensionally and intensionally consistent. However, for $(e_1, y_1) \notin \tilde{I}_1$ in the power context, we can check that $e_1^{\tilde{I}_1 I} \times \pi_\rho^{-1}(y_1^{\tilde{I}_1})^I \subseteq I$ (see Fig. 9). Therefore, $\tilde{I}_1$ is not faithful.

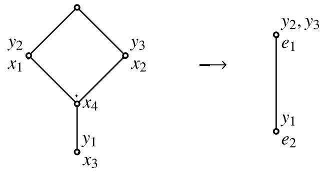

Fig. 9. The base concept lattice (left) and the power concept lattice (right) from Example 4.2.

$$\begin{tikzcd}
\mathfrak{B}_o(X,Y,I) \arrow[r, "\pi_\rho^+|_{\mathfrak{B}_o(X,Y,I)}"] \arrow[d, shift left, "(\ )^I|_{\mathfrak{B}_o(X,Y,I)}"] & \wp(X/\rho) \arrow[r, "(\ )^{\tilde{I}_1}"] & \mathfrak{B}_a(X/\rho, Y/\theta, \tilde{I}_1) \arrow[d, shift left, "(\ )^{\tilde{I}_1}|_{\mathfrak{B}_a(X/\rho,Y/\theta,\tilde{I}_1)}"] \\
\mathfrak{B}_a(X,Y,I) \arrow[r, "\pi_\theta^+|_{\mathfrak{B}_a(X,Y,I)}"] \arrow[u, shift left, "(\ )^I|_{\mathfrak{B}_a(X,Y,I)}"] & \wp(Y/\theta) \arrow[r, "(\ )^{\tilde{I}_1}"] & \mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_1) \arrow[u, shift left, "(\ )^{\tilde{I}_1}|_{\mathfrak{B}_o(X/\rho,Y/\theta,\tilde{I}_1)}"]
\end{tikzcd}$$

Fig. 10. Commutative diagram for faithfulness of $\tilde{I}_1$.

Remark 4.1. Proposition 4.1 provides a way to describe the faithful lifted incidence relation by a commutative graph. Note that our approach also use the technique of lifting a given function onto the powerset level [15,27]: given a function $f : X \longrightarrow Y, f^+ : \wp(X) \longrightarrow \wp(Y)$ is defined by $f^+(A) := \{f(x) \mid x \in A\}; f^- : \wp(Y) \longrightarrow \wp(X)$ is defined by $f^-(B) := \{x \mid f(x) \in B\}$. For our purpose, $\pi_\rho^+|_{\mathfrak{B}_o(X,Y,I)}$ (respectively, $\pi_\theta^+|_{\mathfrak{B}_a(X,Y,I)}$) is used to denote the restriction of π_ρ^+ (respectively, π_θ^+) on $\mathfrak{B}_o(X, Y, I)$ (respectively, $\mathfrak{B}_a(X, Y, I)$). Then (π_ρ, π_θ) is concept-preserving if and only if the diagram in Fig. 10 is commutative. By Proposition 4.2, $\tilde{I}_1$ is faithful if and only if the diagram in Fig. 10 is commutative, π_ρ is extensionally continuous, and π_θ is intensionally continuous.

In the following, we will study the properties of faithful lifted incidence relations. Note that given a (ρ, θ)-context (X, Y, I, ρ, θ), for any $x \in X$, both $[x^I]^{\tilde{I}_1}$ and $[x]^{\tilde{I}_1\tilde{I}_1}$ are extents of $(X/\rho, Y/\theta, \tilde{I}_1)$. The following proposition shows that they coincide in the situation of faithfulness.

Proposition 4.2. *If $\tilde{I}_1$ is faithful, then for any $x \in X$ and $y \in Y$,*

(1) $[x^I]^{\tilde{I}_1} = [x]^{\tilde{I}_1\tilde{I}_1} = [x^{II}]^{\tilde{I}_1\tilde{I}_1}$;
(2) $[y^I]^{\tilde{I}_1} = [y]^{\tilde{I}_1\tilde{I}_1} = [y^{II}]^{\tilde{I}_1\tilde{I}_1}$.

Proof. (1): Suppose $x \in X$. We first prove $[x^I]^{\tilde{I}_1} = [x]^{\tilde{I}_1\tilde{I}_1}$. By Proposition 2.6, $[x^I] \subseteq [x]^{\tilde{I}_1}$ and thus $[x]^{\tilde{I}_1\tilde{I}_1} \subseteq [x^I]^{\tilde{I}_1}$. Conversely, let x_0 be any element of X with $[x_0] \in [x^I]^{\tilde{I}_1}$ and y_0 any element of Y with $[y_0] \in [x]^{\tilde{I}_1}$. Since $[x^I] \subseteq [x]^{\tilde{I}_1}$, it holds that $[y_0] \in [x^I]$ or $[y_0] \in [x]^{\tilde{I}_1} \setminus [x^I]$. If $[y_0] \in [x^I]$, it is clear that $([x_0], [y_0]) \in \tilde{I}_1$. If $[y_0] \in [x]^{\tilde{I}_1} \setminus [x^I]$, we can also show $([x_0], [y_0]) \in \tilde{I}_1$ as follows: Assume $([x_0], [y_0]) \notin \tilde{I}_1$. As $\tilde{I}_1$ is faithful, there exist $x' \in \pi_\theta^{-1}([x_0]^{\tilde{I}_1})^I$ and $y' \in \pi_\rho^{-1}([y_0]^{\tilde{I}_1})^I$ such that $(x', y') \notin I$. Since $[y_0] \in [x]^{\tilde{I}_1}$, we have $[x] \in [y_0]^{\tilde{I}_1}$ and $x \in \pi_\rho^{-1}([y_0]^{\tilde{I}_1})$. As $y' \in \pi_\rho^{-1}([y_0]^{\tilde{I}_1})^I$, $(x, y') \in I$, i.e., $y' \in x^I$. Since $[x_0] \in [x^I]^{\tilde{I}_1}$, it follows that $([x_0], [y']) \in \tilde{I}_1$, i.e., $[y'] \in [x_0]^{\tilde{I}_1}$. Because $x' \in \pi_\theta^{-1}([x_0]^{\tilde{I}_1})^I$, we have $(x', y') \in I$ which is a contradiction. Now we conclude that $([x_0], [y]) \in \tilde{I}_1$ for any $y \in Y$ with $[y] \in [x]^{\tilde{I}_1}$. This means $[x_0] \in [x]^{\tilde{I}_1\tilde{I}_1}$ and thus $[x^I]^{\tilde{I}_1} \subseteq [x]^{\tilde{I}_1\tilde{I}_1}$. Therefore, we obtain $[x^I]^{\tilde{I}_1} = [x]^{\tilde{I}_1\tilde{I}_1}$.

For $[x]^{\tilde{I}_1\tilde{I}_1} = [x^{II}]^{\tilde{I}_1\tilde{I}_1}$, note that (π_ρ, π_θ) is concept-faithful by Proposition 4.1. Then we have $(\pi_\rho, \pi_\theta)(x^{II}, x^I) = ([x^I]^{\tilde{I}_1}, [x^{II}]^{\tilde{I}_1})$ is a power concept. It follows that $[x]^{\tilde{I}_1\tilde{I}_1} = [x^{II}]^{\tilde{I}_1\tilde{I}_1}$.

Likewise, we have (2). □

Proposition 4.3. *Let $\{A_t \mid t \in T\}$ be any non-empty family of base extents and $\{B_t \mid t \in T\}$ any non-empty family of base intents. If $\tilde{I}_1$ is faithful, then*

(1) $\left[\left(\bigcup_{t\in T} A_t\right)^{II}\right]^{\tilde{I}_1} = \left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1}$;

(2) $\left[\left(\bigcup_{t\in T} B_t\right)^{II}\right]^{\tilde{I}_1} = \left[\bigcup_{t\in T} B_t\right]^{\tilde{I}_1}$.

Proof. We only prove (1). First, it follows from $\bigcup_{t\in T} A_t \subseteq \left(\bigcup_{t\in T} A_t\right)^{II}$ that $\left[\left(\bigcup_{t\in T} A_t\right)^{II}\right]^{\tilde{I}_1} \subseteq \left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1}$. Conversely, assume that $\left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1} \subseteq \left[\left(\bigcup_{t\in T} A_t\right)^{II}\right]^{\tilde{I}_1}$ does not hold. Then there exist $x_0 \in X$ and $y_0 \in Y$ such that $x_0 \in \left(\bigcup_{t\in T} A_t\right)^{II}$, $[y_0] \in \left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1}$, and $([x_0], [y_0]) \notin \tilde{I}_1$. By Definition 4.1, there exist $x' \in \pi_\theta^{-1}([x_0]^{\tilde{I}_1})^I$ and $y' \in \pi_\rho^{-1}([y_0]^{\tilde{I}_1})^I$ such that $(x', y') \notin I$. Note that $[y_0] \in \left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1}$, we have $\left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1\tilde{I}_1} \subseteq [y_0]^{\tilde{I}_1}$ which implies that $\bigcup_{t\in T} A_t \subseteq \pi_\rho^{-1}([y_0]^{\tilde{I}_1})$. Thus, we have

$\pi_\rho^{-1}([y_0]^{\tilde{I}_1})^I \subseteq \left(\bigcup_{t\in T} A_t\right)^I$ and so $y' \in \left(\bigcup_{t\in T} A_t\right)^I$. Since $x_0 \in \left(\bigcup_{t\in T} A_t\right)^{II}$, we have $(x_0, y') \in I$, which entails $[y'] \in [x_0]^{\tilde{I}_1}$. Since $x' \in \pi_\theta^{-1}([x_0]^{\tilde{I}_1})^I$, $(x', y') \in I$ since we know that $(x', y') \notin I$. Therefore, $\left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1} \subseteq \left[\left(\bigcup_{t\in T} A_t\right)^{II}\right]^{\tilde{I}_1}$ and thus $\left[\left(\bigcup_{t\in T} A_t\right)^{II}\right]^{\tilde{I}_1} = \left[\bigcup_{t\in T} A_t\right]^{\tilde{I}_1}$.

(2) can be analogously proved. □

The following proposition illustrates that the extents and intents of power context are closed under appropriate transformations if $\tilde{I}_1$ is faithful.

Proposition 4.4. *If $\tilde{I}_1$ is faithful, then for any extent α and intent β of $(X/\rho, Y/\theta, \tilde{I}_1)$,*

(1) $[\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1} = \alpha$;
(2) $[\pi_\theta^{-1}(\beta)^I]^{\tilde{I}_1} = \beta$.

Proof. (1): From Proposition 2.7, $[\pi_\rho^{-1}(\alpha)^I] \subseteq [\pi_\theta^{-1}(\alpha^{\tilde{I}_1})] = \alpha^{\tilde{I}_1}$. Since α is an extent of $(X/\rho, Y/\theta, \tilde{I}_1)$, $\alpha = \alpha^{\tilde{I}_1\tilde{I}_1} \subseteq [\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1}$. Conversely, assume that $[\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1} \subseteq \alpha$ is not valid. Then there exists $x \in X$ such that $[x] \in [\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1}$ and $[x] \notin \alpha$. Since α is an extent of $(X/\rho, Y/\theta, \tilde{I}_1)$, there exists $y \in Y$ such that $[y] \in \alpha^{\tilde{I}_1}$ with $([x], [y]) \notin \tilde{I}_1$. As $\tilde{I}_1$ is faithful, by Definition 4.1, there exist $x' \in \pi_\theta^{-1}([x]^{\tilde{I}_1})^I$ and $y' \in \pi_\rho^{-1}([y]^{\tilde{I}_1})^I$ such that $(x', y') \notin I$. As $\alpha \subseteq [y]^{\tilde{I}_1}$, it follows that $\pi_\rho^{-1}([y]^{\tilde{I}_1})^I \subseteq \pi_\rho^{-1}(\alpha)^I$ and thus $y' \in \pi_\rho^{-1}(\alpha)^I$. Since $[x] \in [\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1}$, we have $([x], [y']) \in \tilde{I}_1$, i.e., $[y'] \in [x]^{\tilde{I}_1}$. Since $x' \in \pi_\theta^{-1}([x]^{\tilde{I}_1})^I$, we have $(x', y') \in I$ which is a contradiction to $(x', y') \notin I$. This implies that $[\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1} \subseteq \alpha$. Therefore, $[\pi_\rho^{-1}(\alpha)^I]^{\tilde{I}_1} = \alpha$.

(2): It can be proved similarly. □

With respect to a faithful lifted incidence relation, either extensional continuity of π_ρ or intensional continuity of π_θ can lead to an order-embedding of $\mathfrak{B}_o(X/\rho, Y/\theta, \tilde{I}_1)$ into $\mathfrak{B}_o(X, Y, I)$. However, based on the concept-faithfulness of (π_ρ, π_θ), we can establish a stronger homomorphism theorem. In fact, a mapping $h : \mathfrak{B}(X, Y, I) \to \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$ can be defined by

$$h(A, B) := ([B]^{\tilde{I}_1}, [A]^{\tilde{I}_1})$$

for any concept (A, B) of (X, Y, I).

From concept-preservation of (π_ρ, π_θ), h is well-defined. It is easy to verify that h is order-preserving. Furthermore, we have

Proposition 4.5. *h is a surjective complete homomorphism. Moreover, h maps object concepts onto object concepts and attribute concepts onto attribute concepts.*

Proof. Let $\{(A_t, B_t) \mid t \in T\}$ be any non-empty family of base concepts. Note that $h\left(\bigwedge_{t\in T}(A_t, B_t)\right) = h\left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)^{II}\right) = \left(\left[\left(\bigcup_{t\in T} B_t\right)^{II}\right]^{\tilde{I}_1}, \left[\bigcap_{t\in T} A_t\right]^{\tilde{I}_1}\right)$ and $\bigwedge_{t\in T} h(A_t, B_t) = \bigwedge_{t\in T}([B_t]^{\tilde{I}_1}, [A_t]^{\tilde{I}_1}) = \left(\bigcap_{t\in T}[B_t]^{\tilde{I}_1}, \left(\bigcup_{t\in T}[A_t]^{\tilde{I}_1}\right)^{\tilde{I}_1\tilde{I}_1}\right) = \left(\left[\bigcup_{t\in T} B_t\right]^{\tilde{I}_1}, \left(\bigcup_{t\in T}[A_t]^{\tilde{I}_1}\right)^{\tilde{I}_1\tilde{I}_1}\right)$. By Proposition 4.3, we have $\left[\left(\bigcup_{t\in T} B_t\right)^{II}\right]^{\tilde{I}_1} = \left[\bigcup_{t\in T} B_t\right]^{\tilde{I}_1}$. It follows that h is $\wedge$-preserving. Similarly, it can be shown that h is $\vee$-preserving.

Next we show that h maps object concepts onto object concepts and attribute concepts onto attribute concepts. Suppose $x \in X$. Notice that $[x^I]^{\tilde{I}_1} = [x]^{\tilde{I}_1\tilde{I}_1}$ by Proposition 4.2(1), it follows that $h(x^{II}, x^I) = ([x^I]^{\tilde{I}_1}, [x^{II}]^{\tilde{I}_1}) = ([x]^{\tilde{I}_1\tilde{I}_1}, [x]^{\tilde{I}_1})$. To show h maps attribute concepts onto attribute concepts, note that for any $y \in Y$, $[y^{II}]^{\tilde{I}_1} = [y]^{\tilde{I}_1}$ by Proposition 4.2(2). Hence $h(y^I, y^{II}) = ([y^{II}]^{\tilde{I}_1}, [y^I]^{\tilde{I}_1}) = ([y]^{\tilde{I}_1}, [y]^{\tilde{I}_1\tilde{I}_1})$.

Finally, we show that h is surjective. Suppose $(\alpha, \beta) \in \mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$. Since α is an extent with respect to $\tilde{I}_1$, there exists a subset $B \subseteq Y$ such that $\alpha = [B]^{\tilde{I}_1} = \bigwedge\{[y]^{\tilde{I}_1} | y \in B\}$ and $(\alpha, \beta) = \bigwedge\{([y]^{\tilde{I}_1}, [y]^{\tilde{I}_1\tilde{I}_1}) \mid y \in B\}$. As h maps attribute concepts onto attribute concepts, we have $(\alpha, \beta) = \bigwedge\{h(y^I, y^{II}) \mid y \in B\}$. Because h is $\wedge$-preserving, it follows that $(\alpha, \beta) = h(\bigwedge\{(y^I, y^{II}) \mid y \in B\}) = h(B^I, B^{II})$. □

Based on the complete homomorphism h, we can define a kernel relation on $\mathfrak{B}(X, Y, I)$ as

$$\ker(h) := \{(c, c') \mid c, c' \in \mathfrak{B}(X, Y, I) \text{ and } h(c) = h(c')\}.$$

From the concept-preservation of (π_ρ, π_θ), we know that $(c, c') \in \ker(h)$ if and only if $[B_1]^{\tilde{I}_1} = [B_2]^{\tilde{I}_1}$ (equivalently, $[A_1]^{\tilde{I}_1} = [A_2]^{\tilde{I}_1}$) for any two base concepts $c = (A_1, B_1)$ and $c' = (A_2, B_2)$. By Theorem 1.3 and Proposition 4.5, $\ker(h)$ is a complete congruence relation on $\mathfrak{B}(X, Y, I)$ and we immediately have the following result.

Theorem 4.1. *Let (X, Y, I, ρ, θ) be a (ρ, θ)-context. If $\tilde{I}_1$ is faithful, then* $\ker(h)$ *is a complete congruence on* $\mathfrak{B}(X, Y, I)$, *and the factor lattice* $\mathfrak{B}(X, Y, I)/\ker(h)$ *is order-isomorphic to* $\mathfrak{B}(X/\rho, Y/\theta, \tilde{I}_1)$.

Theorem 4.1 indicates that if the self-relations on objects and attributes can guarantee the faithfulness of the lifted incidence relation, the base concepts can be partitioned into pair-wise disjoint groups. This may provide an efficient way to classify the concepts of a context into interval-like groups via accommodating appropriate self-relations into the setting of the original context.

5. Research community revisited

As indicated in the Introduction, a straightforward application of FCA is insufficient for the purpose of modeling research communities as formal concepts. The framework of power concepts provides a suitable approach that makes it possible to reuse much of the traditional theory of FCA for characterizing research communities. The proposed "research community as concept" approach [28,29] takes advantage of the co-authorship relation among researchers. From the point of view of social network analysis, grouping authors into network components based on the co-authorship relation is aligned with the "School of Thought" idea, which not only reduces context size, but also provides an avenue for a useful interplay between the author–venue cross-domain relation and the co-authorship self-relation.

The bulk of this paper has been directed toward the intriguing question of how base contexts (lattices) are related to the power contexts (lattices). As demonstrated by Corollary 2.1 and Proposition 2.9, one perspective is manifested in the observation that every power concept corresponds to an interval of base concepts. The notions of extensional consistency and intensional consistency provides another perspective of the co-authorship relation when using power concepts to identify research communities, as "intervals" representing possible ranges of topics represented by a research community. Due to the independence, or the decoupling, of the author–venue relation and the co-authorship relation, the relationship between the base concept lattice and the power concept lattice becomes rather complex. The existences of Galois connections provide a handle to explore the structural relationships in a systematic way. As shown in Corollaries 3.1 and 3.2, these Galois connections provide a translation between the research communities and research units. More specifically, one research community (represented formally by a power concept) might be approximated by a collection of research units (represented formally by base concepts). Furthermore, from the results of Section 4 we can see that if the co-authorship relation is special enough (formally, the corresponding lifted incidence relation is faithful), the research communities obtained by the power concepts are all independent because they do not share any research units with each other.

In information retrieval, we are sometimes interested in specific research communities [23]. An unresolved challenge is how to automatically or semi-automatically generate a complete venue list that represents a research area (or community). The power concept analysis framework provides several possibilities that allow one to combine and compare venue lists from the baseline author-venue context and the lifted "school-venue" context. Proposition 2.6 and the subsequent discussion suggest how the resulting venue lists might be related, with one being potentially overly restrictive, and the other likely overly broad. Thus the study presented in this paper should be regarded as the initiation of the power concept and applications, rather than final words on this topic.

6. Conclusions and future work

Our study is motivated by application of power concepts in case study of research communities. The specific theoretical development has been aimed at structural relationship between the base concept lattice and its power concept lattice, in hopes of informing the understanding of the variety of power concept lattices. We focus our investigation on the fundamental connections between the base context (concept lattice) and the power context (concept lattice). Particularly, canonical Galois connections have been established under the conditions of extensional consistency and intensional consistency. We have further introduced the notion of faithfulness to a specific lifted incidence relation that corresponds to the case study of research communities. In this case, we have shown that the power concept lattice is order-isomorphic to a factor lattice of the base concept lattice.

Several intriguing topics remain. First, the background self-relations on objects or attributes may be more specific in some practical applications. Additionally, other different types of liftings besides those given in Definition 2.2 might be required for practical purposes. Therefore, a more complete picture of the power concept analysis framework is desirable. Second, as approaches have been proposed to fuzzify classical construction of concept lattice [6,3,14,19,21], it would be valuable to investigate the fuzzification of the power concepts framework. This may widen the scope of applicability, particularly for identifying research communities where real-world co-authorship relationship is not binary, but based on numeric data and measurements that will be subjected to intensive preprocessing before the approach of power concept analysis is applied. Third, based on recent private communication, power contexts and their lattices seem to be related to the notion of *annotated ordered set* developed by Cliff Joslyn and his collaborators [12]. We leave this a future work to explore the applications of power contexts in the structural analysis of biomedical ontologies such as the Gene Ontology. Fourth, a challenge in FCA-based applications is to develop efficient algorithms for constructing concept lattices [24,25]. Our approach, especially the notion of faithfulness in Section 4, may also provide a method in the divide-and-conquer style to improve the efficiency of existing algorithms. Finally, because the relationships or flows between nodes in the sense of social network analysis can be modeled as (fuzzy) relations [16], our approach may provide a basic framework for exploring the combination of social network analysis methods.

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